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THE SPACING OF ORIFICES FOR THE MEASUREMENT OF PRESSURE DISTRIBUTIONS

By Max M. Munk

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THE SPACING OF ORIFICES FOR THE MEASUREMENT OF PRESSURE DISTRIBUTIONS.

By Max M. Munk.

Summary

The following report has been prepared for publication by the National Advisory Committee for Aeronautics. Suitable locations of orifices for the measurement of pressure distributions have been discussed. Tables are given for quickly laying out these locations and for quickly and easily computing the resultant air forces from the result of the measurements.

Introduction

For many aeronautical problems, the mechanical interaction between adjacent particles of air or between particles of air and an adjacent rigid boundary can be assumed exactly enough to be a pressure. This means that the three shear components become zero, and that the three remaining compression components become equal to a pressure, say p. The physical dimension of a pressure is $\frac{\text{force}}{\text{area}}$, and since with a pure and genuine pressure the force is always directed at right angle to its reference area, a pressure cannot be said to have or to occupy any specific direction - it is not a "vector."

The pressure of moving air is generally different at different points. It has to be realized that the pressure now under discussion really exists; it is the actual force per unit area between adjacent particles. There is, however, no method known to measure an absolute pressure as such. Only the difference between two pressures can be measured. It is, therefore, necessary to establish a standard pressure before entering into the discussion of any numerical relation. Such standard or zero pressure is chosen differently in different cases. When discussing heavier-than-air craft (neglecting any buoyancy of the air) it is customary and most convenient to consider as zero pressure the pressure of the atmosphere at the same altitude, when at rest, that is, in absence of the airplane and of any wind. This standard pressure is not constant under the ordinary assumptions of mechanics, but is different at different altitudes. On the other hand, when discussing the buoyancy of air as with airships, the variation of the pressure of resting air is of greatest importance and the standard pressure has to be chosen otherwise.

In the following, we denote as pressure the difference between the actual pressure at any point and the standard pressure as defined above. This actual pressure, constituting the interaction between adjacent particles is often called "static pressure." Actual pressure and static pressure are identically the same.

Near the surfaces of solids moving through air, the shear forces between adjacent particles are particularly large, but they can still be practically neglected in so far as the compressive stress of the air can be assumed the same at a specific point in whatever direction of the reference plane. We assume the existence of a pressure distribution, of a distribution of actual or static pressure, over the surface of an aircraft or of a portion thereof. The present discussion relates to the measurement of this pressure distribution by means of small orifices, distributed over the surface under investigation. Each orifice is connected to a manometrical device. It is assumed that the pressure at the orifice is not affected by the presence of the orifice, and that the pressure is correctly recorded or indicated by the manometer.

If sufficient orifices are provided, the test gives information about the pressure distribution over the surface investigated. This information can be made as complete as desired by simply increasing the number of orifices. In most cases, the information about the pressure distribution is used for the computation of one or several components of the resultant air force equivalent to the pressure distribution measured. This resultant air force is not always the entire resultant air force as the shear forces cannot be measured by means of pressure orifices, and hence the resultant of the shear forces is not contained in the resultant of the pressures.

The component of the resultant air force is obtained by projecting each surface element parallel to the component. Each projection is then multiplied by its pressure and all products summed up. Often the surface is closed, and there are always pairs of surface elements having their projection in common. The contribution of such pair of surface elements is equal to the product of the projection and of the difference of the pressures of the two surface elements. This difference or sum of pressure of a group of surface elements situated at a straight line parallel to the component of resultant force will be called resultant pressure. A resultant pressure has the same dimension as an ordinary pressure $\frac{\text{force}}{\text{area}}$. It is, however, distinguished by a direction, the direction of the component of the resultant force. Hence the resultant pressure is a vector.

General Considerations Governing the Spacing of the Pressure Orifices

I proceed to discuss those problems connected with pressure distribution measurements that are specially related to the chief purpose of such measurements, namely, the determination of the resultant air force. These problems are chiefly of a mathematical character and indeed very attractive to the speculative mind. They subordinate themselves to the two practical questions:

a) How should the pressure holes be distributed?

b) How can the resultant force be conveniently obtained from the observed pressures?

The second question is independent of the first one. In many cases, the tests are prepared without giving full thought to the mode of distributing the pressure holes, or, a systematic and well-proportioned distribution cannot be used on account of details of the structural arrangement of the aircraft. The choice of the distribution of the pressure holes, on the other hand, should be taken so as (a) to obtain the most exact resultant air force with a given number of holes, (b) to require the least number of orifices for a desired degree of exactness, (c) to be able to determine the integral with the least possible amount of time and labor, and of errors involved in the method of computation; all that as far as can be done practically.

When choosing the distribution of the pressure holes there should also be taken into account the type of pressure distribution to be expected if such previous knowledge exists. For the areas of high resultant pressure contribute comparatively much to the resultant air force, and since further the errors of the instruments and the slope of the pressure curves are large within such range, the pressure holes should be spaced closer at such regions in order to obtain the greatest exactness of the integral. Otherwise spoken, the pressure holes may have a different "weight," to borrow an expression from the calculus of probability. The allotment of these weights requires much judgment and experience. A general discussion, not referring to a particular distribution of the weight, is of little value.

When distributing the pressure holes, the investigator should also keep in mind the end of the research, that is, to obtain general information on the subject enabling him to predict to some extent the pressure distribution over another object, different but of similar type. It becomes always necessary to select a family of curves, the pressure distribution of which is a substitute for the pressure distribution over the entire area, and to select a finite number of pressure holes along each curve to be a substitute for the pressure distribution over this curve. Now, it is often possible to specify the curves in such a way that the pressure distribution along them becomes particularly simple, or at least approaches a particularly simple distribution. Often the points can be so located as to form at the same time two families of curves, each of them with a different type of pressure distribution, but simpler than the pressure distribution along any third curve. It is impossible to lay down general rules for such proceeding, but the investigator will learn to follow these suggestions when he has become accustomed not to overlook this side of the preparation of the test.

Spacing Along a Line With Constant Weight and Without Structural Restrictions.

To begin with the simplest problem, I suppose for the present all pressure holes to have the same weight, and their choice to be left open to the critical mind without limitation by the structural arrangement of the parts of the aircraft. Let further the family of curves be chosen, and for the present, the attention be concentrated to one curve only. Pressure holes of equal weight are to be distributed along one curve with the purpose of obtaining the pressure distribution or the distribution of resultant pressure, but chiefly of obtaining their integral, the resultant air force or a part of it. After what has been said in the first section of this paper, it is supposed that the position of each pressure hole is given by its normal projection on a plane perpendicular to the desired air force component. This projection of the curve, moreover, may be assumed at present to be a straight line, in order to make the discussion as simple and plain as possible.

The pressures being determined empirically, they do not follow any simple mathematical law, or if they do, the law is not known to the investigator. We have, therefore, arrived at the problem to integrate a function empirically given at a finite number of points and hence we resort to the so-called mechanical or numerical integration, as opposed to the integration by analytical methods. However, the discussion of such integration as found in most text books on mathematical analysis does not strictly refer to our problem. In case of measuring a pressure distribution, there are the values of the pressure given at a finite number of values of the abscissa only, rather than at all points along the range of integration. Geometrically expressed, a finite number of points rather than a curve is given. Neither the complete pressure curve nor its derivatives at any point is known. Even this is said too much. It cannot even be strictly said that, at a finite number of points the pressure is given, for the pressures observed are naturally distorted by experimental errors. It would carry me too far out of the reach of my immediate topic to dwell on the theory of such errors, and to discuss the methods of determining the most probable errors and the most probable results of the tests. The methods as generally taught are directly applicable to the determination of pressure distributions and of the resulting air force therefrom. The existence of experimental errors has been mentioned here only because of its bearing on the choice of the location of the pressure holes, and on the method of integrating the pressure. The integration of the pressure has to be made in such a way that no experimental errors be given an undue influence on the final results of the integration. The errors are distributed in an unknown way, but subject to the general laws of probability. The mode of integration should be such that the probable error of the integral stands in a due relation to the single error of each reading. This now will be the case, and the best results will be obtained if each reading enters into the process of integration with a weight as nearly equal as possible to its real weight. This general remark will become clearer when we proceed to the different methods worked out.

The distributions of the pressure holes, on the other hand, if systematically chosen, are based on the method of integration and hence are closely connected with the last consideration. All other things being equal, a good distribution of pressure holes leads to such methods of obtaining the final integral, which give each reading its proper weight.

Almost the same demands follow from the condition of smallest errors of numerical computation. Such numerical computations (if any) consist necessarily of repeated additions and multiplications, and each single step is closely connected with the choice of the distribution of the pressure holes. A distribution of pressure holes is poor, if it involves taking small differences of large quantities. The error of the resultant force is smallest when all resultant pressures are of equal sign and uniformly distributed. It is in this case that the errors of computation should become smallest too. They will, if all pressures enter with nearly their true weight into the integration, not if they are first to be multiplied by multipliers of greatly varying magnitude, or by such even varying in sign. This point of view will immediately be taken up in the next section.

Graphical Integration

Let there now be n points, on a straight line, at which the resultant pressure is to be measured. Let the line extend from $x = a$ to $x = b$; let the points at which the pressure has been determined have the abscissae x_1 to x_n, and let the pressures at these points be denoted as p_1 to p_n. Suppose for simplicity of expression that the pressures be plotted as ordinates giving n pressure points (Fig. 1).

The determination of the resultant force (for a strip of the width = 1, say) requires two steps: (a) all pressure points have to be connected by a curve (mentally or actually) and (b) this curve has to be integrated.

Up to now it has been almost general practice to perform these steps graphically. The pressure points were connected by an arbitrary curve subjected, however, to the condition that it appeared "smooth" to the artistic feeling of the draftsman. Analyzing this condition closer, it consists chiefly in the mathematical condition that the value of the ordinate of the curve, of the slope of the curve, and at best of the curvature should not vary abruptly. Now, the last condition, though it may be in keeping with the actual pressure distribution, does not by itself necessarily lead to the most exact value of the integral. The graphical method in itself is not particularly inexact. On the contrary, it can be made as exact as desired and as is possible in view of the errors of the pressure readings by using large enough diagrams. In some cases, the graphical method is the most convenient one too, in particular, if the spacing of the pressure holes had to be or was irregular, and if not, many pressure distributions at the same holes are measured, making it otherwise necessary to work out with much pains an inconvenient scheme of numerical integration, to be applied a few times only. Here, then, the method of least mental work is at the same time the method of least work. Even then, however, the graphical method possesses one distinct and important disadvantage. The curves between the pressure points, whether actually drawn in, or whether only the mental illustration of a mathematical process, are not known and therefore arbitrary to some extent. The way of choosing them has an appreciable effect on the integration. Hence if two tests are repeated, or only the evaluation of one test, the results will be different in general. It is not possible, or at least it never has been worked out, to draw the connections according to some standard scheme.

I wish to emphasize the fact that the distribution of the pressure holes has an equal effect on the exactness of the result in either of the two cases where the integration has been made graphically or numerically. He is mistaken, therefore, who thinks that the intention to integrate graphically relieves him from the duty to carefully select the spacing of the pressure holes. The discussion of the spacings farther below refers in the same way to all tests, no matter how the resulting pressure is intended to be determined. The choice of an unsystematic spacing without external reasons therefore always deserves censure. For this reason, the cases are infrequent where a graphical integration is recommended. The general procedure of the graphical integration is generally known. I wish to make only one remark. It often occurs that the pressure curve intersects with the base line, the pressure being alternately positive and negative. Even then, working with the ordinary planimeter, it is allowed to circumscribe the pressure area one time, following first the entire pressure curve and closing it along the base line.

The same remark holds true when determining mechanically the static moment or the moment of inertia with respect to a point of the base line. The instrument used is slightly different from an ordinary planimeter. But again the entire pressure area has to be circumscribed, and again it is unnecessary to split this area in parts of equal sign. Follow first the entire pressure curve and then the base line.

The numerical integration of the pressure in most cases and in some respects in all cases is superior to the graphical integration. There are different numerical methods again, with advantages and disadvantages peculiar to each of them. It is easy to select one of them as a standard method and thus to obtain always the same integral from the same test data. A greater consistency of the results and safer conclusions are gained.

Numerical Integration, Cotesius' Method

In most cases the pressure holes can be systematically spaced. Then the numerical integration is decidedly easier and less time-absorbing than the graphical integration. It becomes quite unnecessary to plot the pressure readings, or at least, if such plots are desired for illustration, they can be made less exact. The mechanical integrators can be dispensed with, and this means an immense saving in time, in labpr, in mental strain and in annoyance. The numerical operations, taking the place of operating a planimeter or a similar instrument, can be chosen to be of the simplest kind. The multiplications can be done exactly enough by means of a slide rule, or more conveniently by means of a good calculation machine; the additions should be made with a calculation machine. It is worthy of remark that these recommendations are in keeping with the general development of performing technical computations. The last century was the century of graphical methods, the wages were then low and the calculation machines bad and expensive. Now, through the development of good calculation machines, all that has changed. The graphical methods are more and more abandoned or only retained for illustrative purposes. In its stead the use of the calculation machine becomes prevalent. And it can generally be said that the calculation machine and the methods based on its use have in common with other machines that which holds generally for the replacement of hand labor by machine labor. The graphical methods are more general and do require less preparation for a novel case and then require less time and less mental work. But for a standard problem, once the scheme for numerical computations has been worked out, the numerical method is easier, less toilsome, less time-absorbing, giving more exact results and giving uniform results, more easily checked and last but not least, does not involve any personal factor. This latter means that any one obtains the same results from the same data, once the method has been decided upon.

I proceed now to the discussion of the different cases. A spacing of pressure holes, which is often found and indeed suggests itself the most readily, is the division of the straight line into equal parts. Let the number of points be n, and hence the number of spaces be $n - 1$. This disagreement between the number of points and number of spaces destroys the uniformity of the arrangement and makes the equal spacing little recommendable, as we shall immediately see.

There have been several methods devised for the numerical integration. It is to be desired that all these methods are in some way independent of the size of the ordinates. By this I mean the following: The computation has to be prepared by computing tables of the numerical values used in the numerical integration. The computation of these tables is laborious, and it is desired that such tables be made once for all, not new ones for each integration. Therefore, the procedure has to consist in the combination of the ordinates with the figures of the integration table by means of simple algebraic operations, This most general case will be treated first. Afterwards I shall take up certain special cases where the particular type of the pressure distribution to be integrated will be taken into account, and yet the methods remain general enough and can be used for all possible values of the ordinates.

An old method of numerical integration is the one of Cotesius.* He chose as curve connecting all n pressure points the algebraic curve of nth degree containing all these points. It is known that there always exists one and only one such curve. This way of connecting the points will be found again farther below, when we discuss the method of Gauss and the improved methods derived therefrom. Now, it must be borne in mind that the addition of expressions of nth degree, say,

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* Roger Cotesius, English mathematician, 1682-1716.

$$f_1\ (x) = y_1 = A_o + A_1 x + A_2 x^2 + \ldots\ldots\ldots + A_n x^n$$

$$f_2\ (x) = y_2 = B_o + B_1 x + B_2 x^2 + \ldots\ldots\ldots + B_n x^n$$

etc., gives a new expression of nth (or lower) degree again. Herein the "A" and "B" denote constants, x denotes the abscissa, and y the ordinate (pressures in this case). On the other hand, it is always possible to write down an expression of nth degree which is zero at all points, x_1, x_2 .... x_n except one, x_m, at which latter it assumes the magnitude of unity. Such expression can be written in the shape

$$f_m(x) = \frac{\varphi_m(x)}{\varphi_m(x_m)} \qquad (1)$$

where φ_m again is

$$\varphi_m = (x-x_1)\ (x-x_2)\ldots(x-x_{m-1})\ (x-x_{m+1})\ldots(x-x_n) \qquad (2)$$

φ_m is the product of $(n - 1)$ factors. At all points x_1 to x_n except at x_m, one of the factors becomes zero, and hence $\varphi\,(x)$ becomes zero. At $x = x_m$, $\varphi_m(x) = \varphi_m(x_m)$ and hence $f_m(x_m)$ becomes 1. (1) is therefore the desired expression of the nth degree, for it will be realized that $f_m(x_m)$ is a constant. Hence the polynomial expressing the curve of nth degree through all points can be written

$$F = f_1\ y_1 + f_2\ y_2 + \ldots\ldots\ldots + f_n\ y_n \qquad (3)$$

where the y denotes the pressure at the point. Indeed, at any point x_m all f except f_m are zero, $f_m = 1$ and hence $F = y_m$.

Now, it is not necessary to actually form F. Let us suppose that (3) be now integrated with respect to dx. Then

$$\int_a^b F dx = y_1 \int_a^b f_1\,dx + y_2 \int_a^b f_2\,dx + \ldots\ldots\; y_n \int_a^b f_n dx \tag{4}$$

It is generally assumed that the length of the base $(b - a) = 1$ (sometimes 2). The multipliers are generally computed for this case.

Hence we have arrived at the following result, applying to any spacing, equal or not:

In order to obtain the integral of the curve of nth degree passing through all n points given, we have to multiply each pressure y_m by a multiplier, say H_m, which multiplier is independent of the value of the pressures y, but only depends on the spacing of the pressure points. The magnitude of the multipliers is

$$H_m = \int_a^b \frac{\varphi_m(x)}{\varphi_m(x_m)}\,dx \tag{5}$$

where φ_m is given by equation (2).

The method can be followed whether the pressure changes its sign in the interval considered or not, and whether the spaces are equal or not.

Cotesius considered only a spacing $(x_2 - x_1)=(x_3 - x_2)$; $x_1 = a$; $x_n = b$; etc., and employed the method described. He was the first to publish a table of the multipliers, H, for a number of points $n = 1$ to $n = 11$. This table is reproduced as Table I in this paper. For reason of symmetry, it is sufficient to give only half of the factors, since

$$H_m = H_{n-m} \tag{6}$$

The desired integral is

$$(x_n - x_1)\ H_1\ y_1 + H_2\ y_2 + \ldots\ldots + H_n\ y_n \tag{7}$$

The inspection of Cotesius' table shows a characteristic of Cotesius' multipliers H which could not easily be anticipated at first approaching the problem. Since the multipliers depend on the spacing only, and the spacing is constant, it would not seem unlikely that the multipliers H become uniformly distributed along the interval, in that H has the largest value in the middle and gradually falls off to the ends of the interval. Such is by no means the case. On the contrary, not only are the differences of two adjacent multipliers of varying sign, but even the multipliers themselves are of varying sign, some of them becoming negative for odd n.

TABLE I

Table of Cotesius

n	H_1	H_2	H_3	H_4	H_5	H_6
2	$\frac{1}{2}$	$\frac{1}{2}$				
3	$\frac{1}{6}$	$\frac{4}{6}$				
4	$\frac{1}{8}$	$\frac{3}{8}$				
5	$\frac{7}{90}$	$\frac{32}{90}$	$\frac{12}{90}$			
6	$\frac{19}{288}$	$\frac{75}{288}$	$\frac{50}{288}$			
7	$\frac{41}{840}$	$\frac{216}{840}$	$\frac{27}{840}$	$\frac{272}{840}$		
8	$\frac{751}{17,280}$	$\frac{3,577}{17,280}$	$\frac{1,323}{17,280}$	$\frac{2,989}{17,280}$		
9	$\frac{989}{28,350}$	$\frac{5,888}{28,350}$	$\frac{-928}{28,350}$	$\frac{10,496}{28,350}$	$\frac{-4,540}{28,350}$	
10	$\frac{2,857}{89,600}$	$\frac{15,741}{89,600}$	$\frac{1,080}{89,600}$	$\frac{19,344}{89,600}$	$\frac{5,778}{89,600}$	
11	$\frac{16,067}{598,752}$	$\frac{106,300}{598,752}$	$\frac{-48,525}{598,752}$	$\frac{272,400}{598,752}$	$\frac{-260,550}{598,752}$	$\frac{427,368}{598,752}$

$x_n - x_1 = 1$

This characteristic of the multipliers for equal spacing is in contradiction to the condition of not deviating too far from the weight of the observed pressures, as discussed before. It really makes the method impractical, and Cotesius' multipliers are seldom used except for very small n. Cotesius' method has some merits for the computation of mathematical tables, where the ordinates are not distorted by errors. It has to be discarded, however, for the integration of empirical observations as the pressure at equally spaced points.

Simpson's* Rule and Generalizations Thereof

In its stead, the so-called Simpson's rule has found a wide application. It refers to an odd n only, which is a distinct disadvantage. Simpson divides the intervals of integration into $\frac{n-1}{2}$ parts. Each interval thus obtained is equally spaced into two parts, and Cotesius' Table for $n = 3$ is applied to it. Cotesius' multipliers for $n = 3$ are in the ratio 1 : 4.1. Adding, now, all integrals for the $\frac{n-1}{2}$ parts of the intervals, the multipliers for the ends of adjacent parts have to be added. Hence Simpson's multipliers are in the ratio

$n = 3$ 1 : 4 : 1 (Like Cotesius)

$n = 5$ 1 : 4 : 2 : 4 : 1

$n = 7$ 1 : 4 : 2 : 4 : 2 : 4 : 1

etc.

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* Thomas Simpson, English mathematician, 1710-1761.

The integral is

$$\frac{a}{3}\,(y_1 + 4y_2 + 2y_3 + 4y_4 + \ldots\ldots)$$

where a denotes the space between two adjacent points.

Now, the imperfectness of Simpson's rule, namely, that the adjacent factors differ in the ratio of $1:2$, is the direct consequence of treating each pair of adjacent points unsymmetrically. The entire base is divided into parts equal to two spaces, and the ordinates at the ends of a space are treated differently, according to whether they are located at the end or in the middle of a double space. Accordingly, we can hope to improve the method by treating all ordinates alike. This can be done by considering each space by itself, not the spaces in pairs.

The area over one space can be computed in first approximation from the values of the ordinates at its ends, closing the space by a straight line connecting the pressure points at the ends of each space (Fig. 4). This procedure means a repeated application of Cotesius' method at each space with $n = 2$. It would result in multipliers standing in a ratio $1 : 2 : 2 \ldots\ldots 2 : 1$. The probability of a reasonable exactness of such integration is not great enough, however. I would not recommend the general employment of this method for the computation of the resultant force from the measurement of pressures at a series of points equally spaced.

Simpson improved the integral over one space by taking into account not only the pressures at the ends, but also the pressures at one more point, namely, the point next to the one at one end of the space in question. It is this unsymmetric proceeding that gives rise to the lack of uniformity of Simpson's multipliers. Hence the logical step to improve Simpson's rule is to take into account additional pressures at points located symmetrically with respect to the interval to be integrated. It is equally logical to choose the number of these additional points as 2, or of all points as 4, this being the smallest number of points admitting of a symmetrical arrangement. And at last there is only one obvious way to select the two additional points; they are the two adjacent to the ends, as shown in Fig. 5. This can always be done except for the last space in the interval. I propose using only one adjacent point for the integration of the end interval, thus using Simpson's rule for the two end intervals only. If there are only two intervals, or three points, Simpson's rule and mine reduce to Cotesius multipliers for $n = 3$.

I proceed to the determination of the multipliers and integrate first over one interval not at the end. Let, for instance, the length of 3 spaces be 2, and let the ends be located at the points $x = +1$ and $x = -1$. Now, suppose a curve of 3d degree to be drawn such that the ordinate y becomes zero at the points $x = -1$, $-\frac{1}{3}$, and $+1$, and $y = 1$ at the point $x=\frac{1}{3}$.

The equation of this curve is

$$y = + \frac{(x^2 - 1)\ (x + \frac{1}{3})}{\left(\left(\frac{1}{3}\right)^2 - 1\right)\ \left(+\frac{1}{3} + \frac{1}{3}\right)}$$

By inserting $x = \pm 1$ or $\pm\frac{1}{3}$, it becomes evident that this curve agrees with the conditions laid down. Now integrate y between $-\frac{1}{3}$ and $+\frac{1}{3}$

$$\int_{-1/3}^{+1/3} \frac{(x^2 - 1)\ \left(x + \frac{1}{3}\right)}{\left(\left(\frac{1}{3}\right)^2 - 1\right)\ \frac{2}{3}}\, dx = \frac{13}{36}$$

It can directly be seen that the value of this integral would have become $\left(\frac{13}{36}\, y\right)$ in case y would have been y_3 instead of 1 at the point $x = + \frac{1}{3}$. For the curve to be integrated would have agreed with the one actually integrated, except for the constant factor y_3. The area is therefore $\frac{13}{36}\, y_3$, or written as the product of the value y_3, the base length $c\left(= \frac{2}{3}\right)$ and a constant factor, the integral is

$$c\ y_3\ \frac{13}{24}$$

The symmetry of the problem shows further, that the integral of a similar curve, having the ordinate y_2 at the point $-\frac{1}{3} \times \frac{3c}{2}$ would be

$$c\ y_2\ \frac{13}{24}$$

The superposition of the two curves gives one having the ordinates zero at the two adjacent points and having the ordinates y_2 and y_3 at the two ends of the integration.

It is possible to compute the multipliers for the adjacent points 1 and 3 in a like way. They follow much more simply from the consideration that for equal ordinates $y_1 = y_2 = y_3 = y_4$ the area must come out $c\ y_1$, and that the two multipliers for the adjacent points must be equal for reason of symmetry. Hence, counting now all four curves, we obtain one passing through points with ordinates y_1 to y_4 at $x = \pm 1.5\ c$, $\pm 0.5\ c$. The integral of this curve, along the middle interval is

$$-\frac{1}{24}\ y_1\ c + \frac{13}{24}\ y_2\ c + \frac{13}{24}\ y_3\ c - \frac{1}{24}\ y_4\ c$$

I pass now to an end interval. Let the base extend from -1 to $+1$, and the integral from 0 to $+1$. The parabola with $y_1 = y_2 = 0$, $y_2 = 1$ has the equation $y = (1 - x^2)$ the integral of which is

$$\int_0^1 (1 - x^2)\ dx = \frac{2}{3}.$$

Now, $c = 1$ and hence the area becomes

$$y_2\ \frac{8}{12}\ c$$

Likewise, let the parabola have the ordinates $y_1 = y_2 = 0$, $y_3 = 1$, giving the equation $y_1 = \frac{1}{2}\ (x^2 + x)$ and the integral

$$\frac{1}{2} \int_0^1 (x^2 + x)\ dx = + \frac{1}{2} \times \frac{5}{6} = + \frac{5}{12}$$

The area is

$$-\frac{1}{12}\ y_3\ c$$

Finally, let $y_3 = 1, \; y_2 = y_0 = 0,$

$$y = \frac{1}{2}\,(x^2 - x)$$

$$\frac{1}{2}\int_0^1 (x^2 - x)\;dx = -\frac{1}{12}\,y_1\,c$$

hence, the general formula for the integrals.

$$\text{Area} = c\left(+\frac{5}{12}\,y_3 + \frac{8}{12}\,y_2 - \frac{1}{12}\,y_1\right).$$

As expected, this gives $y_1\,c$ in case that $y_1 = y_2 = y_3$.

I am now enabled to write down the multipliers for the general case of n points with equal spacing, by adding the integrals over all single intervals.

For instance, for $n = 6$

1st interval	5/12	8/12	-1/12			
2d "	-1/24	13/24	13/24	-1/24		
3d "		-1/24	13/24	13/24	-1/24	
4th "			-1/24	13/24	13/24	-1/24
5th "				-1/12	8/12	5/12
Sum	9/24	28/24	23/24	23/24	28/24	9/24
	1	2	3	4	5	6

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TABLE II

Improved Multipliers H for Equal Spacing.

n = 2	(Cotesius)	$H_1 = \frac{1}{2}$,	$H_2 = \frac{1}{2}$						
n = 3	(Cotesius and Simpson)	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{1}{3}$					
n = 4	$\frac{3}{8}$	$\frac{9}{8}$	$\frac{9}{8}$	$\frac{3}{8}$					
n = 5	$\frac{9}{24}$	$\frac{28}{24}$	$\frac{22}{24}$	$\frac{28}{24}$	$\frac{9}{24}$				
n = 6	$\frac{9}{24}$	$\frac{28}{24}$	$\frac{23}{24}$	$\frac{23}{24}$	$\frac{28}{24}$	$\frac{9}{24}$			
n = 7	$\frac{9}{24}$	$\frac{28}{24}$	$\frac{23}{24}$	$\frac{24}{24}$	$\frac{23}{24}$	$\frac{28}{24}$	$\frac{9}{24}$		
n = n	$\frac{9}{24}$	$\frac{28}{24}$	$\frac{23}{24}$	$\frac{24}{24}$ ...	$\frac{24}{24}$ ...	$\frac{24}{24}$	$\frac{23}{24}$	$\frac{28}{24}$	$\frac{9}{24}$

$$\text{Area} = c\,(H_1\,y_1 + H_2\,y_2 + \cdot\cdot\cdot\cdot\cdot\cdot\cdot + H_n\,y_n)$$

The preceding table is to be used in the same way as Simpson's; the ordinates are to be multiplied by their respective multiplier, the products so obtained are to be added together and the sum has to be multiplied by the length of one space. This holds, no matter what the signs of the ordinates are, but of course a negative ordinate gives a negative product and this sign has to be given attention when adding up all products.

Multipliers for Arbitrary Spacing

The foregoing method can be extended to an arbitrary spacing. Again, a parabola of third degree can be imagined to be drawn through four consecutive points and the area of the middle section be obtained by integration. I omit the simple computation and give only the result. Let a, b, c be the length of three consecutive spaces. Then the area over the middle space b becomes

$$\text{Area} = \frac{b}{12}\,\frac{b^2 + 2bc + 4ab + 6ac}{(c + b)\,a}\,y_2 + \frac{b}{12}\,\frac{b^2 + 2ab + 4bc + 6ac}{(a + b)\,c}\,y_3$$

$$- \frac{b}{12}\,\frac{b^3 + 2ab^2}{(a + b + c)\,(b + c)\,c}\,y_4 - \frac{b}{12}\,\frac{b^3 + 2b^2\,c}{(a + b + c)\,(b + a)\,a} \qquad (8)$$

The first and last sections are integrated by choosing the parabola through the first three or last three points as boundary line.

The area results

$$\frac{ay_1}{6}\,\frac{(2a + 3b)}{(a + b)} + y_2\,\frac{a(a + 3b)}{6b} - \frac{a^3\;y_3}{6b\,(a + b)} \qquad (9)$$

Each of the single areas has to be expressed by means of these formulas (8) and (9) as sums of the ordinates y, each multiplied by a numerical constant, and all these expressions have to be added, giving in each case a series of multipliers to be used as in all cases discussed before.

The computation of the multipliers, though by no means difficult, is laborious, and it is always preferable to distribute the pressure holes systematically and to use standard tables rather than tables made up for the special occasion.

Gauss' Method of Integration

Proceeding now to unequal but systematic spacing, there has first to be mentioned the method of Gauss.* This method has the immense advantage of a uniform variation of the multipliers. This uniformity will also be maintained for all distributions derived from Gauss' method.

Gauss himself stressed chiefly the point of highest accuracy with a given number of ordinates in conjunction with the methods not directly depending on the values of the ordinates. The general method is quite analogous to Cotesius' method and the other methods discussed. Each ordinate is multiplied by its multipliers, which latter depends on the spacing only. All products are then added. The multipliers are again computed by integrating the curve of $(n - 1)$th degree having the ordinates $y_1 = y_2 = - - - y_{m-1} = y_{m+1} - - - - - - - = y_n = 0$ and $y_m = 1$.

The variation left consists therefore in the spacing of the ordinates used. Gauss thought chiefly of the integration of mathematical functions and supposed them to be well approximated by a series of terms of powers of x,

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* Karl Friedrich Gauss, German mathematician, 1777-1855.

$$A_0 + A_1 x + A_2 x^2 + \text{etc.} \tag{10}$$

Now, if there would only be n terms in the expression (10) every method of integration of this kind would be absolutely exact, since then the curve of nth degree would coincide with the curve to be integrated. Gauss proposes to select the spacing in such a way that the integration would also be exact for the next n terms of the series (10). A short reflection will show that this is equivalent to the condition that all curves passing through the base points and being of a degree not higher than n have the integral zero. Their addition to any curve would not change the ordinates in question and hence would not change the integral. This gives n equations for the computation of the n abscissae.

Gauss has computed the abscissae and multipliers up to $n = 7$. In Table III, these values are reproduced and further the values for $n = 8$ to $n = 12$, as computed for this paper by Dr. Paul E. Hemke, an American mathematician and member of the technical staff of the National Advisory Committee for Aeronautics.

TABLE III

Gauss' Table and P. E. Hemke's Table*

Base length = 1

	$n = 1$	
$X_1 = 0.5$		$H_1 = 1$
	$n = 2$	
$X_1 = 0.21682$		$H_1 = 0.5$
	$n = 3$	
$X_1 = 0.11270$		$H_1 = 5/18 = 0.2777$
$X_2 = 0.5$		$H_2 = 8/18 = 0.4444$
	$n = 4$	
$X_1 = 0.11943$		$H_1 = 0.17393$
$X_2 = 0.33009$		$H_2 = 0.32607$
	$n = 5$	
$X_1 = 0.04691$		$H_1 = 0.11846$
$X_2 = 0.23077$		$H_2 = 0.23931$
$X_3 = 0.50000$		$H_3 = 0.28444$
$X_4 = 0.76923$		$H_4 = 0.23931$
$X_5 = 0.95307$		$H_5 = 0.11846$
	$n = 6$	
$X_1 = 0.033765$		$H_1 = 0.085662$
$X_2 = 0.16940$		$H_2 = 0.18038$
$X_3 = 0.38069$		$H_3 = 0.23396$
$X_4 = 0.61931$ .381		$H_4 = 0.23396$
$X_5 = 0.83060$ .169		$H_5 = 0.18038$
$X_6 = 0.96623$ .034		$H_6 = 0.085662$

Table III (Cont.)

n = 7

X_1 = 0.025446	H_1 = 0.064742
X_2 = 0.12923	H_2 = 0.13985
X_3 = 0.29708	H_3 = 0.19092
X_4 = 0.50000	H_4 = 0.20898
X_5 = 0.70292	H_5 = 0.19092
X_6 = 0.87077	H_6 = 0.13985
X_7 = 0.97455	H_7 = 0.064742

n = 8

X_1 = 0.019855	H_1 = 0.050614
X_2 = 0.10137	H_2 = 0.11119
X_3 = 0.23723	H_3 = 0.15685
X_4 = 0.40828	H_4 = 0.18134
X_5 = 0.59172	H_5 = 0.18134
X_6 = 0.76277	H_6 = 0.15685
X_7 = 0.89833	H_7 = 0.11119
X_8 = 0.98015	H_8 = 0.050614

n = 9

X_1 = 0.015920	H_1 = 0.040637
X_2 = 0.081985	H_2 = 0.090324
X_3 = 0.19331	H_3 = 0.13031
X_4 = 0.33787	H_4 = 0.15617
X_5 = 0.50000	H_5 = 0.16512
X_6 = 0.66213	H_6 = 0.15617
X_7 = 0.80669	H_7 = 0.13031
X_8 = 0.91802	H_8 = 0.090324
X_9 = 0.98408	H_9 = 0.040637

Table III (Cont.)

n = 10

$X_1 = 0.013047$	$H_1 = 0.033336$
$X_2 = 0.067469$	$H_2 = 0.074729$
$X_3 = 0.16030$	$H_3 = 0.10954$
$X_4 = 0.28330$	$H_4 = 0.13463$
$X_5 = 0.42556$	$H_5 = 0.14776$
$X_6 = 0.57444$	$H_6 = 0.14776$
$X_7 = 0.71670$	$H_7 = 0.13463$
$X_8 = 0.83971$	$H_8 = 0.10954$
$X_9 = 0.93253$	$H_9 = 0.074729$
$X_{10} = 0.98695$	$H_{10} = 0.033336$

n = 11

$X_1 = 0.010886$	$H_1 = 0.027839$
$X_2 = 0.056469$	$H_2 = 0.062795$
$X_3 = 0.13492$	$H_3 = 0.093150$
$X_4 = 0.24045$	$H_4 = 0.11658$
$X_5 = 0.36523$	$H_5 = 0.13141$
$X_6 = 0.50000$	$H_6 = 0.13646$
$X_7 = 0.63477$	$H_7 = 0.13141$
$X_8 = 0.75955$	$H_8 = 0.11658$
$X_9 = 0.86508$	$H_9 = 0.093150$
$X_{10} = 0.94353$	$H_{10} = 0.062795$
$X_{11} = 0.98911$	$H_{11} = 0.027839$

Table III (Cont.)

$n = 12$

$X_1 = 0.0092195$	$H_1 = 0.023588$
$X_2 = 0.047942$	$H_2 = 0.053470$
$X_3 = 0.11505$	$H_3 = 0.080039$
$X_4 = 0.20634$	$H_4 = 0.10158$
$X_5 = 0.31608$	$H_5 = 0.11675$
$X_6 = 0.43738$	$H_6 = 0.12457$
$X_7 = 0.56262$	$H_7 = 0.12457$
$X_8 = 0.68392$	$H_8 = 0.11675$
$X_9 = 0.79366$	$H_9 = 0.10158$
$X_{10} = 0.88495$	$H_{10} = 0.080039$
$X_{11} = 0.95206$	$H_{11} = 0.053470$
$X_{12} = 0.99078$	$H_{12} = 0.023588$

Taking up now the question of exactness, it seems sound to expect a small probable error from Gauss' method. The question cannot be answered directly, as integrating curves determined by single points that are empirically found is quite another thing than integrating a mathematical function. No safe criterion can be given in the former case except that one can discuss a more or less probable exactness. It should, however, be borne in mind that even for the integration of mathematical functions, Gauss' method is exact for $2n$ terms in powers of x. The power series, however, is by no means the only one, nor in any way particularly distinguished from an expansion of a function into a series progressing in terms of some other functions. For any kind of expansion we can compute a distribution of points such that the integration is exact for $2n$ terms of such series. In general, this gives distributions different from Gauss'. As a consequence, the multipliers will be different, too. No criterion has come to my knowledge deciding which of such expansions gives the most exact results, and probably this question cannot be answered at all but depends on the function to be integrated. We have arrived at a probability problem of a very general kind.

Since the powers of x (parabolas) have played the most important part in modern mathematics, it will probably be wisest to follow so great and eminent a man as Karl Friederich Gauss, and to adopt his method for the general case as the most exact one. A slight variation does not produce any large difference of the result any way. Besides, the chief advantage is the uniformity of the multipliers rather than the large exactness.

It should now be clearly understood that Gauss recommended this method under the condition that the probability of the magnitude of the ordinates be equal along the entire range of integration. In our case, this would be the case if the resultant pressure is measured along a line over a limited portion of the surface of the aircraft, not extending to the edge or end, at which the resulting pressure is primarily zero. Moreover, nothing would be known beforehand about the type of pressure distribution to be expected. In such cases, Gauss' (and Hemke's) spacing and multipliers can be used without modification whatsoever.

Spacing for Special Types of Pressure Distribution

In many cases, where the line reaches up to an edge of the aircraft, or through it (as the wing chord, or the diameter of a round airship hull, for instance), the resulting pressure is sure to be zero at the two ends of the line. This case is so general with investigation of pressure distribution that it is worth while to consider it separately. It pays to modify the Gauss table for this case. The proceeding is somewhat arbitrary. Two methods suggest themselves at first glance, the first of which will seem inferior to the second at closer examination.

In the first instance, Gauss' method could be generalized in such a way that two abscissae are given; in this case two zero ends. n other points are to be computed so that the integration becomes exact for n additional forms of the power series.

I prefer another way, which leads to different results. The problem as stated just before involves only a vanishing resultant pressure at the ends, but it does not include the resultant pressure to be small near these ends. I prefer selecting a probability or weight function along the entire range, giving zero at the ends. A convenient weight function is $\cos \pi \frac{x}{2}$. Let the base extend between the points $x = \pm 1$ and p denote the pressure. Then write

$$p = \cos \pi \frac{x}{2} \varphi(x) \qquad \Phi(x) = \frac{p}{\cos \pi \frac{x}{2}}$$

We have to integrate

$$\int_{-1}^{+1} p dx = \int_{-1}^{+1} \Phi(x) \cos \pi \frac{x}{2} \, dx = \frac{2}{\pi} \int_{-1}^{+1} \Phi(x) \, d \left(\sin \pi \frac{x}{2}\right)$$

I insert now $\sin \pi \frac{x}{2}$ for x in Gauss' tables, obtaining new abscissae. The multipliers with respect to $\Phi(x)$ remain unaltered. But since $\varphi(x) \sin \pi \frac{x}{2}$ are measured; the multipliers are equal to $H = \frac{2 H_0}{\pi \sin \pi \frac{x}{2}}$ and can then directly be applied to the values of the pressures, that is, the new x_0 and H are to be used in the same way as before. Table IV gives the modified Gauss-Hemke Table, for the length 2 of the base.

They are computed by Dr. Hemke for this paper.

TABLE IV.

$n = 5$

$X_1 = -0.72202$	$H_1 = 0.35666$
$X_2 = -0.36200$	$H_2 = 0.36160$
$X_3 = 0$	$H_3 = 0.36217$
$X_4 = 0.36200$	$H_4 = 0.36160$
$X_5 = 0.72202$	$H_5 = 0.35666$

Table IV (Cont.)

n = 6

$X_1 = -0.76470$	$H_1 = 0.30191$
$X_2 = -0.45992$	$H_2 = 0.30614$
$X_3 = -0.15338$	$H_3 = 0.30674$
$X_4 = 0.15338$	$H_4 = 0.30674$
$X_5 = 0.45992$	$H_5 = 0.30614$
$X_6 = 0.76470$	$H_6 = 0.30191$

n = 7

$X_1 = -0.79602$	$H_1 = 0.26172$
$X_2 = -0.53180$	$H_2 = 0.26540$
$X_3 = -0.26604$	$H_3 = 0.26597$
$X_4 = 0$	$H_4 = 0.26608$
$X_5 = 0.26604$	$H_5 = 0.26597$
$X_6 = 0.53180$	$H_6 = 0.26540$
$X_7 = 0.79602$	$H_7 = 0.26172$

n = 8

$X_1 = -0.82000$	$H_1 = 0.23099$
$X_2 = -0.58682$	$H_2 = 0.23423$
$X_3 = -0.35226$	$H_3 = 0.23474$
$X_4 = -0.11744$	$H_4 = 0.23488$
$X_5 = 0.11744$	$H_5 = 0.23488$
$X_6 = 0.35226$	$H_6 = 0.23474$
$X_7 = 0.58682$	$H_7 = 0.23423$
$X_8 = 0.83000$	$H_8 = 0.23099$

Table IV (Cont.)

n = 9

$X_1 = -0.83891$	$H_1 = 0.20668$
$X_2 = -0.63026$	$H_2 = 0.20984$
$X_3 = -0.42038$	$H_3 = 0.21007$
$X_4 = -0.21022$	$H_4 = 0.21021$
$X_5 = 0$	$H_5 = 0.21024$
$X_6 = 0.21022$	$H_6 = 0.21021$
$X_7 = 0.42038$	$H_7 = 0.21007$
$X_8 = 0.63026$	$H_8 = 0.20984$
$X_9 = 0.83891$	$H_9 = 0.20668$

n = 10

$X_1 = -0.85427$	$H_1 = 0.18706$
$X_2 = -0.66544$	$H_2 = 0.18966$
$X_3 = -0.47552$	$H_3 = 0.19008$
$X_4 = -0.38648$	$H_4 = 0.19022$
$X_5 = -0.09512$	$H_5 = 0.19026$
$X_6 = 0.09512$	$H_6 = 0.19026$
$X_7 = 0.28538$	$H_7 = 0.19022$
$X_8 = 0.47552$	$H_8 = 0.19008$
$X_9 = 0.66544$	$H_9 = 0.18966$
$X_{10} = 0.85427$	$H_{10} = 0.18706$

n = 11

$X_1 = -0.86691$	$H_1 = 0.17077$
$X_2 = -0.69450$	$H_2 = 0.17317$
$X_3 = -0.52110$	$H_3 = 0.17354$

Table IV (Cont.)

$n = 11$

$X_4 = -0.34746$	$H_4 = 0.17365$
$X_5 = -0.17374$	$H_5 = 0.17373$
$X_6 = 0$	$H_6 = 0.17375$
$X_7 = 0.17374$	$H_7 = 0.17373$
$X_8 = 0.34746$	$H_8 = 0.17365$
$X_9 = 0.52110$	$H_9 = 0.17354$
$X_{10} = 0.69450$	$H_{10} = 0.17317$
$X_{11} = 0.86691$	$H_{11} = 0.17077$

$n = 12$

$X_1 = -0.87757$	$H_1 = 0.15714$
$X_2 = -0.71894$	$H_2 = 0.15933$
$X_3 = -0.55939$	$H_3 = 0.15969$
$X_4 = -0.39964$	$H_4 = 0.15981$
$X_5 = -0.23980$	$H_5 = 0.15985$
$X_6 = -0.07994$	$H_6 = 0.15987$
$X_7 = 0.07994$	$H_7 = 0.15987$
$X_8 = 0.23980$	$H_8 = 0.15985$
$X_9 = 0.39964$	$H_9 = 0.15981$
$X_{10} = 0.55939$	$H_{10} = 0.15969$
$X_{11} = 0.71894$	$H_{11} = 0.15933$
$X_{12} = 0.87757$	$H_{12} = 0.15714$

A second modification of the Gauss table refers to the measurement of the resultant pressure distribution along a wing chord. It is known that the resultant pressure distribution along a chord is chiefly concentrated near the leading edge with most practical sections.

A weight function $(\sin \pi x \cos \frac{\pi x}{2})$ was chosen by Dr. Hemke. The procedure leads to Table V, likewise computed by Dr. Hemke. The leading edge is at the side where the spacing is narrow, that is, $x = 1$, the trailing edge being at $x = 0$.

TABLE V.

Base length = 1.

n = 5

$X_1 = 0.34630$	$H_1 = 0.29473$
$X_2 = 0.59533$	$H_2 = 0.21273$
$X_3 = 0.77938$	$H_3 = 0.15658$
$X_4 = 0.90889$	$H_4 = 0.10208$
$X_5 = 0.98253$	$H_5 = 0.044724$

n = 6

$X_1 = 0.31001$	$H_1 = 0.26449$
$X_2 = 0.53536$	$H_2 = 0.19516$
$X_3 = 0.70820$	$H_3 = 0.15212$
$X_4 = 0.84088$ .159	$H_4 = 0.11330$
$X_5 = 0.93448$ .066	$H_5 = 0.073617$
$X_6 = 0.98748$ .013	$H_6 = 0.032094$

n = 7

$X_1 = 0.28190$	$H_1 = 0.24083$
$X_2 = 0.48609$	$H_2 = 0.18000$
$X_3 = 0.64963$	$H_3 = 0.14479$

Table V (Cont.)

n = 7

$X_4 = 0.77938$		$H_4 = 0.11504$
$X_5 = 0.87986$		$H_5 = 0.085816$
$X_6 = 0.95066$		$H_6 = 0.055561$
$X_7 = 0.99059$		$H_7 = 0.024140$

n = 8

$X_1 = 0.25938$	743	$H_1 = 0.22176$
$X_2 = 0.44934$	.551	$H_2 = 0.16709$
$X_3 = 0.60103$	.399	$H_3 = 0.13701$
$X_4 = 0.72576$	.274	$H_4 = 0.11294$
$X_5 = 0.82733$	.173	$H_5 = 0.090232$
$X_6 = 0.90612$	.094	$H_6 = 0.067232$
$X_7 = 0.96151$	.038	$H_7 = 0.043400$
$X_8 = 0.99266$	.007	$H_8 = 0.018812$

n = 9

$X_1 = 0.24086$		$H_1 = 0.20604$
$X_2 = 0.41818$		$H_2 = 0.15607$
$X_3 = 0.56015$		$H_3 = 0.12960$
$X_4 = 0.67934$		$H_4 = 0.10932$
$X_5 = 0.77938$		$H_5 = 0.090897$
$X_6 = 0.86120$		$H_6 = 0.072705$
$X_7 = 0.92464$		$H_7 = 0.054079$
$X_8 = 0.96915$		$H_8 = 0.034827$
$X_9 = 0.99413$		$H_9 = 0.015070$

Table V (Cont.)

n = 10

$X_1 = 0.22532$	.775	$H_1 = 0.19281$
$X_2 = 0.39149$	.609	$H_2 = 0.14658$
$X_3 = 0.52533$	.475	$H_3 = 0.12278$
$X_4 = 0.63903$	.361	$H_4 = 0.10518$
$X_5 = 0.73639$	.264	$H_5 = 0.089719$
$X_6 = 0.81866$	.181	$H_6 = 0.074835$
$X_7 = 0.88602$	.114	$H_7 = 0.059835$
$X_8 = 0.93818$	.061	$H_8 = 0.044430$
$X_9 = 0.97472$	025	$H_9 = 0.028560$
$X_{10} = 0.99518$	.005	$H_{10} = 0.013343$

n = 11

$X_1 = 0.21208$	$H_1 = 0.18153$
$X_2 = 0.36865$	$H_2 = 0.13337$
$X_3 = 0.49531$	$H_3 = 0.11661$
$X_4 = 0.60381$	$H_4 = 0.10097$
$X_5 = 0.69802$	$H_5 = 0.087653$
$X_6 = 0.77938$	$H_6 = 0.075122$
$X_7 = 0.84831$	$H_7 = 0.062717$
$X_8 = 0.90475$	$H_8 = 0.050091$
$X_9 = 0.94638$	$H_9 = 0.037145$
$X_{10} = 0.97890$	$H_{10} = 0.023842$
$X_{11} = 0.99598$	$H_{11} = 0.010295$

Table V (Cont.)

$n = 12$

$X_1 = 0.20060$	$H_1 = 0.17171$
$X_2 = 0.34884$	$H_2 = 0.13115$
$X_3 = 0.46914$	$H_3 = 0.11104$
$X_4 = 0.57283$	$H_4 = 0.096930$
$X_5 = 0.66378$	$H_5 = 0.085179$
$X_6 = 0.74351$	$H_6 = 0.074384$
$X_7 = 0.81263$	$H_7 = 0.063880$
$X_8 = 0.87125$	$H_8 = 0.053326$
$X_9 = 0.91922$	$H_9 = 0.042556$
$X_{10} = 0.95627$	$H_{10} = 0.031508$
$X_{11} = 0.95627$?, .98127	$H_{11} = 0.020199$
$X_{12} = 0.99681$	$H_{12} = 0.0087146$

Tables for Pressure Distribution Around a Circle

A particular distribution of the abscissae, neither constant nor derived from Gauss' rule, is the projection of points equally spaced around a circle with the base as diameter. This occurs when measuring the pressure distribution over the surface of a round airship hull. For reasons of symmetry, only 4, 8, 12, 16, etc., points are of interest. The base can pass one of the points or be symmetrical to two points of intersecting points.

Dr. Hemke has computed the multipliers for these two cases under the assumption that at the ends the resultant pressure is zero. The results are given in Tables VI and VII.

TABLE VI.

n = number of points in circle of unit radius.

m = number of projections on the diameter.

x_k = abscissae of projections on diameter.

H_k = multipliers to use in evaluating integrals.

(a) $n = 4, \quad m = 3$

$x_1 = -1 = -x_3$

$x_2 = 0$

$H_1 = H_3 = \frac{1}{3}$

$H_2 = \frac{4}{3}$

(b) $n = 8, \quad m = 5$

$x_1 = -1 = -x_5$

$x_2 = -.70711 = -x_4$

$x_3 = 0$

$H_1 = .06667 = H_5$

$H_2 = .53333 = H_4$

$H_3 = .80000$

(c) $n = 12, \quad m = 7$

$x_1 = -1 = -x_7$

$x_2 = -.86603 = -x_6$

$x_3 = -.50000 = -x_5$

$x_4 = 0$

$H_1 = .02857 = H_7$

$H_2 = .25397 = H_6$

$H_3 = .45714 = H_5$

$H_4 = .52064$

(d) $n = 16, \quad m = 9$

$x_1 = -1 = -x_9$

$x_2 = -.92388 = -x_8$

$x_3 = -.70711 = -x_7$

$x_4 = -.38268 = -x_6$

$x_5 = 0$

$H_1 = .01587 = H_9$

$H_2 = .14622 = H_8$

$H_3 = .27937 = H_7$

$H_4 = .36172 = H_6$

$H_5 = .39364$

TABLE VII.

(a) $n = 4, \quad m = 2$

$x_1 = -.70711 = -x_2$

$H_1 = \frac{4}{3} = H_2$

(b) $n = 8, \quad m = 4$

$x_1 = -.92388 = -x_4$

$x_2 = -.38268 = -x_3$

$H_1 = .34477 = H_5$

$H_3 = .72190 = H_4$

(c) n = 12, m = 6

$$x_1 = -.96593 = -x_6$$

$$x_2 = -.70711 = -x_5$$

$$x_3 = -.25882 = -x_4$$

$$H_1 = .15420 = H_6$$

$$H_2 = .36825 = H_5$$

$$H_3 = .50611 = H_4$$

(d) n = 16, m = 8

$$x_1 = -.98079 = -x_8$$

$$x_2 = -.83147 = -x_7$$

$$x_3 = -.55557 = -x_6$$

$$x_4 = -.19509 = -x_5$$

$$H_1 = .08693 = H_8$$

$$H_2 = .21705 = H_7$$

$$H_3 = .32680 = H_6$$

$$H_4 = .38509 = H_5$$

It is seen that for such distribution the multipliers become uniform again.

Before closing this section, I wish to make one remark on Tchebycheff's* method of integration. He distributes the points so as to obtain the multipliers = 1. This is in use in naval architecture, but not for the measurements indicated in the subject of this paper. We have to perform so many algebraic operations before obtaining the pressure point, that one more

* Tchebycheff, Russian mathematician, 1821-1894.

multiplication does not matter. Greater exactness should not be sacrificed to saving one multiplication per pressure observed.

TABLE VIII.

Tchebycheff's Table.

n					
2	0.5773				
3	0	0.7071			
4	0.1876	0.7947			
5	0	0.3745	0.8325		
6	0.2666	0.4225	0.8662		
7	0	0.3239	0.5797	0.8839	
9	0	0.1697	0.5288	0.6010	0.9116

Conclusion

The tables, given for the integration of the pressure, can also be used for the computation of functions of the observed pressure, as, for instance, for the computation of the static moment of the resultant air force with respect to some axis. The function of each observed pressure rather than the pressure itself has to be multiplied by the multipliers and the products added.

The choice of the family of curves along which the pressure holes are arranged should follow the same rules as just given for the single points. The problems are indeed identical. For instance, the span of a wing can be divided by the use of Table III, and then each chart spaced in accordance with Table IV. If the wing is not rectangular, the area rather than the span can be divided according to Table III.

Similarly, in the airship hulls, circles will be the primary curves, and equal spacing is recommendable because both pitching and yawing will generally be investigated. The axis can be divided again by dividing the lateral projected area according to Table III.

No general rules can be given for other cases, but the investigator should be sufficiently familiar with the principles of integration and of the tables presented in this paper to select the spacing with common sense and with careful judgment. There is often more than one good distribution of pressure orifices; the choice between several good distributions is then a matter of taste and of intuition, and the choice of an unsystematic distribution with no special advantage should not be tolerated.

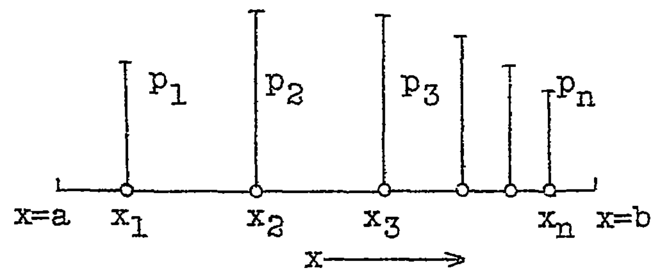

Fig.1

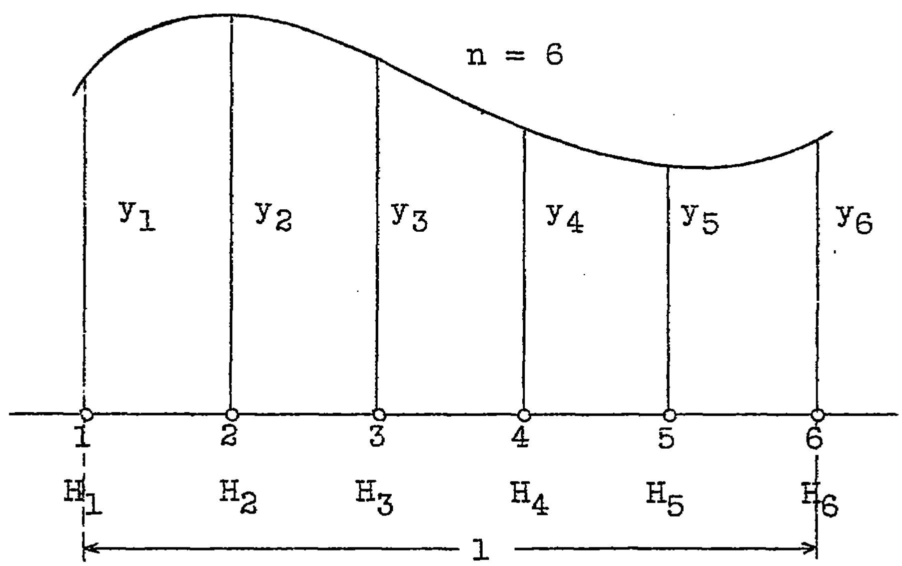

Fig.2

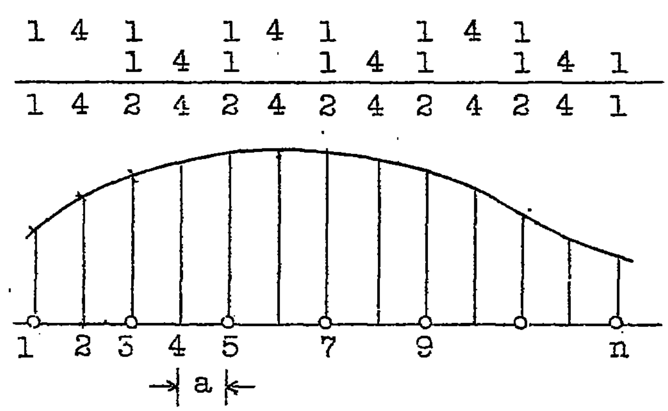

Fig.3

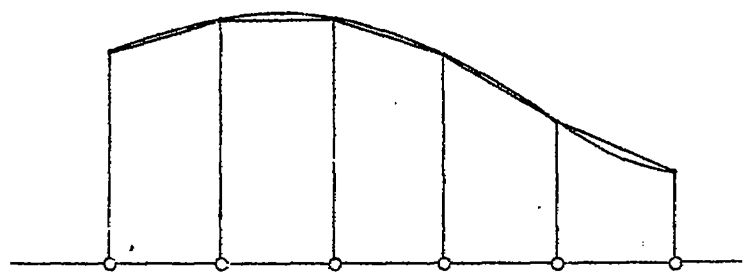

Fig.4

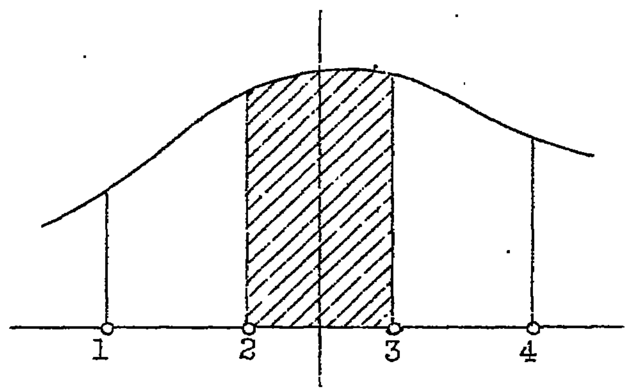

Fig.5

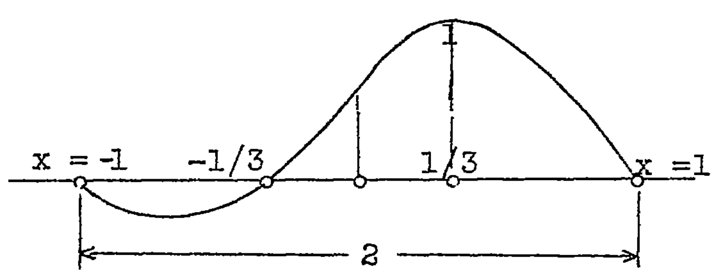

Fig.6

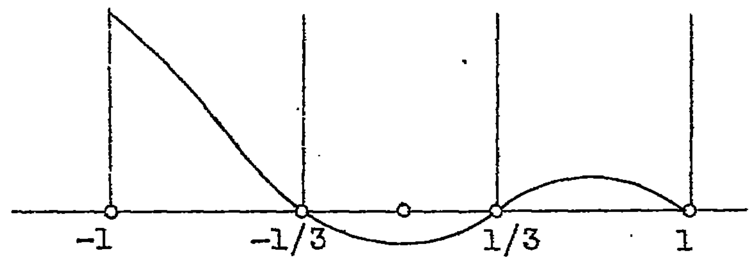

Fig.7

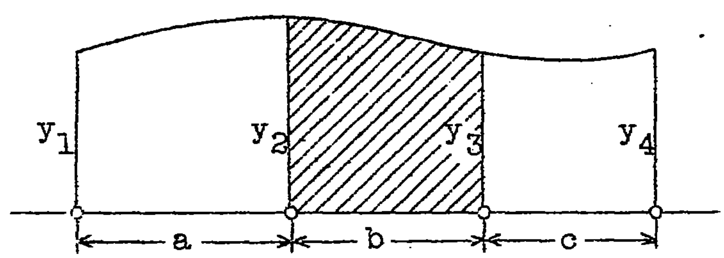

Fig.8

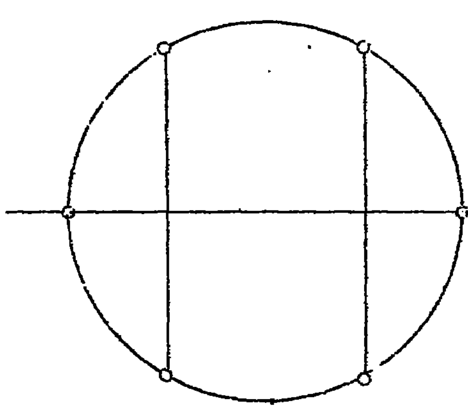

Fig.9

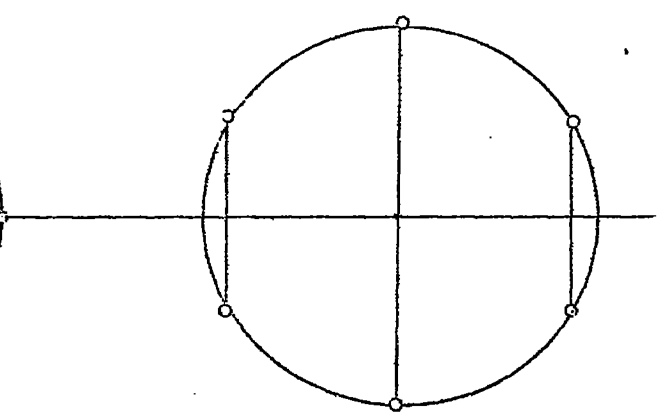

Fig.10